

NASA TT F-10,800

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FACILITY FORM 8	67-25813	
	(ACCESSION NUMBER)	(THRU)
	7	(CODE)
	(PAGES)	32
	(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

Translation of "Pro ruynuvannya plastinki z shchilinoyu"  
Prikladnaya Mekhanika. Institut Mekhaniki AN UkrSSR  
Vol. 7, No. 5, pp. 516-520, 1961

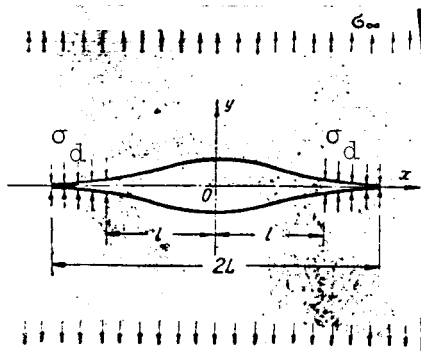
## ON THE FRACTURE OF A PLATE CONTAINING A CRACK

P. M. Vitvitskiy and M. Ya. Leonov (L'vov)

ABSTRACT. A simplified model of a solid body (ref. 1) is used to solve the problem of the fracture of an infinite plate containing a crack during tension by forces perpendicular to the direction of the crack. To determine the critical (fracture) load, the authors obtained formula (19), which can be applied both for ideally brittle materials, as well as for materials in which the fracturing process is accompanied by microplastic deformations. Griffith's formula follows from (19) as a special case. A consequence of formula (19) is that the strength of a plate with a crack of very short length is close to that of a defectless plate, whereas Griffith's formula in this case gives an infinite value for the strength.

1. An infinite plate with a  $2l$ -long crack (see figure) is subjected to continuous tension by forces perpendicular to the direction of the crack. We will determine the level of the tension  $\sigma_\infty$  at which the plate is destroyed.

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Let us assume that the material of the plate meets the following conditions (ref. 1): a) the maximum normal tension does not exceed a certain magnitude  $\sigma_d$  referred to as the

durability threshold of the material, i. e.,

$$\sigma_{\max} \leq \sigma_d; \quad (1)$$

b) the relationship between the tension and the deformations is governed by Hooke's law if the tensile stress is below  $\sigma_d$ ; c) if the

tensile stress reaches a point of deformation which meets the conditions of the linear theory of elasticity and condition (1), cracks will appear in some places (in the area of the weaker bonds); d) the surfaces of such cracks are either drawn to tension  $\sigma_d$ , if the distance between them does not exceed a certain magnitude  $\delta_c$ , or they do not interact at all, if the distance is greater than  $\delta_c$ .

The magnitude  $\delta_c$  for ideally brittle (amorphous) materials is defined by the following formula

$$\delta_c = \frac{2T}{\sigma_d}, \quad (2)$$

\*

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where  $T$  is the surface energy of the material.

The magnitude  $\delta_c$  can be defined experimentally in the case of many materials whose destruction is accompanied by microplastic deformation.

2. We know that no matter how small the load ( $\sigma_\infty$ ) on the plate is, the tension-deformation conditions developing near the ends of the crack are such that condition (1) under Hooke's law is not fulfilled. Depending on the solid-body model in use, the weak zones (fissures) developing at the ends of the crack widen the initial crack in the plate whose surfaces are drawn to the tension areas  $\sigma_d$  (see figure); the contour of the widened crack is thus defined by the /517 following formula

$$X_\nu(x, \pm 0) = 0; Y_\nu(x, \pm 0) = \begin{cases} 0, & |x| < l; \\ \sigma_d, & l \leq |x| \leq L. \end{cases} \quad (3)$$

The length  $2L$  of the widened crack is unknown; it must be defined in such a way as to satisfy condition (1).

We will use M. I. Muskhelishvili's method (ref. 2) to find the solution to the problem. We will find the functions  $\varphi(\zeta)$  and  $\psi(\zeta)$  which are expressed as follows in case of unilateral tension of the plate:

$$\begin{aligned} \varphi(\zeta) &= \frac{1}{8} L \sigma_\infty \zeta + \varphi_0(\zeta), \\ \psi(\zeta) &= \frac{1}{4} L \sigma_\infty \zeta + \psi_0(\zeta). \end{aligned} \quad (4)$$

The holomorphic functions  $\varphi_0(\zeta)$  and  $\psi_0(\zeta)$  of the infinity are defined by the following formulas

$$\varphi_0(\zeta) = -\frac{1}{2\pi i} \int_\gamma \frac{f_0}{\sigma - \zeta} d\sigma, \quad (5)$$

$$\psi_0(\zeta) = -\frac{1}{2\pi i} \int_\gamma \frac{\bar{f}_0}{\sigma - \zeta} d\sigma - \zeta \frac{1 + \zeta^2}{\zeta^2 - 1} \varphi_0'(\zeta),$$

where

$$f_0 = f - \frac{1}{8} L \sigma_\infty \left[ \sigma + \frac{3 - \sigma^2}{\sigma(1 - \sigma^2)} \right]. \quad (6)$$

The preset function  $f$  in this case is characterized by boundary conditions (3):

$$f = \int_{\omega(\sigma_0)}^{\omega(\sigma)} Y(x, \pm 0) dx = \begin{cases} \sigma_d [\omega(\sigma) - l], & -L \leq x \leq l; \\ 0, & |x| < l; \\ \sigma_d [\omega(\sigma) + l], & l \leq x \leq L, \end{cases} \quad (7)$$

where  $\omega(\sigma)$  are the limiting values of function

$$z = \omega(\zeta) = \frac{1}{2} L \left( \zeta + \frac{1}{\zeta} \right), \quad (8)$$

which reflects the plane  $z=x+iy$  with the  $2L$ -long crack on the external part of circle  $\gamma$  in the plane  $\zeta = \rho e^{i\alpha}$ ;  $\sigma = e^{i\alpha}$  is the point on  $\gamma$ ;  $\sigma_0 = e^{i\alpha_0}$  when

$$\alpha_0 = \arccos \frac{l}{L}. \quad (9)$$

Substituting (6), (7) and (8) in (5) and calculating the required integrals by the known formulas, we find functions  $\varphi_0(\zeta)$  and  $\psi_0(\zeta)$  which include certain

invariable items that do not affect the tension-connected deformation conditions, and that will hereafter be disregarded. Knowing these functions and taking (4) into account, we finally obtain

$$\begin{aligned} \varphi(\zeta) &= \frac{1}{8} L \sigma_\infty \left( \zeta - \frac{3}{\zeta} \right) + \frac{\sigma_d}{2\pi i} \left[ \frac{2L}{\zeta} i \alpha_0 - \frac{L}{2} \left( \zeta + \frac{1}{\zeta} \right) \ln \frac{\sigma_0^2 - \zeta^2}{\bar{\sigma}_0^2 - \bar{\zeta}^2} - \right. \\ &\quad \left. - l \ln \frac{(\sigma_0 + \zeta)(\bar{\sigma}_0 - \bar{\zeta})}{(\sigma_0 - \zeta)(\bar{\sigma}_0 + \bar{\zeta})} \right], \\ \psi(\zeta) &= \frac{1}{4} L \sigma_\infty \left( \zeta + \frac{1}{\zeta} - \frac{4\zeta}{\zeta^2 - 1} \right) + \frac{\sigma_d}{2\pi i} \left[ \frac{4L\zeta}{\zeta^2 - 1} i \alpha_0 - \right. \\ &\quad \left. - l \ln \frac{(\sigma_0 + \zeta)(\bar{\sigma}_0 - \bar{\zeta})}{(\sigma_0 - \zeta)(\bar{\sigma}_0 + \bar{\zeta})} \right]. \end{aligned} \quad (10)$$

The tension-connected deformation conditions of the plate are defined by the following formulas (2):

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$$X_x + Y_y = 2[\Phi(\zeta) + \overline{\Phi(\zeta)}], \quad (11)$$

$$Y_y - X_x + 2iX_y = 2 \left[ \frac{\overline{\omega(\zeta)}}{\omega'(\zeta)} \Phi'(\zeta) + \Psi(\zeta) \right],$$

$$\frac{E}{1+\nu} (u + iv) = \frac{3-\nu}{1+\nu} \varphi(\zeta) + \frac{\omega(\zeta)}{\omega'(\zeta)} \overline{\varphi'(\zeta)} - \overline{\psi(\zeta)}, \quad (12)$$

where E is Young's modulus,  $\nu$  is the Poisson coefficient, and  $u$  and  $v$  the travel components on the  $x$  and  $y$  axes, respectively;

$$\Phi(\xi) = \frac{\varphi'(\xi)}{\omega'(\xi)}; \quad \Psi(\xi) = \frac{\psi'(\xi)}{\omega'(\xi)}. \quad (13)$$

Substituting (10) and (8) in (13) and (11), we find in particular

$$Y_y(x, 0)|_{|x|>L} = \sigma_\infty + 2 \left( \sigma_\infty - \frac{2\alpha_0\sigma_d}{\pi} \right) \frac{1}{q^2 - 1} - \frac{2\sigma_d}{\pi} \operatorname{arctg} \frac{\sin 2\alpha_0}{\cos 2\alpha_0 - q^2}, \quad (14)$$

where

$$q = |\xi|_{y=0} = \frac{|x|}{L} + \sqrt{\left(\frac{x}{L}\right)^2 - 1}.$$

With  $|x|=L$ , the denominator in the other invariable of formula (14) equals zero. The fulfillment of condition (1) therefore calls for the assumption that  $\alpha_0 = \frac{\pi\sigma_\infty}{2\sigma_d}$ . Then from formula (9) we find the half-length of the widened crack

$$L = l \sec \frac{\pi\sigma_\infty}{2\sigma_d}. \quad (15)$$

For such a value of  $L$ , the tensions at the ends of the widened crack are continuous, i. e.,  $Y_y(L+0, 0) = Y_y(L-0, 0) = \sigma_d$ , whereby the normal tensions reach the maximum magnitude  $\sigma_d$  only in the area of weakened bonds ( $l \leq |x| \leq L, y=0$ ).

Knowing the value of  $L$ , we will define the tension in the plate on the basis of (8), (10), (11) and (13) by the following formulas

$$\begin{aligned} X_x + Y_y &= \sigma_\infty - \frac{\sigma_d}{\pi i} \times \\ &\times \ln \frac{(l^2 - z^2 + li\sqrt{L^2 - l^2} - z\sqrt{z^2 - L^2})(l^2 - \bar{z}^2 + li\sqrt{L^2 - l^2} - \bar{z}\sqrt{\bar{z}^2 - L^2})}{(l^2 - z^2 - li\sqrt{L^2 - l^2} - z\sqrt{z^2 - L^2})(l^2 - \bar{z}^2 - li\sqrt{L^2 - l^2} - \bar{z}\sqrt{\bar{z}^2 - L^2})}; \\ Y_y - X_x + 2iX_y &= \sigma_\infty + \frac{2\sigma_d(z - \bar{z})l\sqrt{L^2 - l^2}}{\pi(z^2 - l^2)\sqrt{z^2 - L^2}}. \end{aligned} \quad (16) \quad /519$$

In particular, on the actual axis for  $L \leq |x| < \infty$  we have

$$\left. \begin{aligned} Y_y &= \sigma_{\infty} - \frac{2\sigma_d}{\pi} \operatorname{arctg} \frac{l\sqrt{L^2-l^2}}{l^2-x^2-x\sqrt{x^2-L^2}}; \\ X_x &= -\frac{2\sigma_d}{\pi} \operatorname{arctg} \frac{l\sqrt{L^2-l^2}}{l^2-x^2-x\sqrt{x^2-L^2}}; \quad X_y = 0. \end{aligned} \right\} \quad (17)$$

3. After the substitution of (8), (10) and (15) it would be easy to use formula (12) to define the magnitude of  $\delta=2v(\pm l, +0)$  on which points  $(\pm l, +0)$  and  $(\pm l, -0)$  of the opposite crack surfaces will part as a result of the deformation

$$\delta = -\frac{8l}{\pi E} \sigma_d \ln \cos \frac{\pi \sigma_{\infty}}{2\sigma_d}. \quad (18)$$

If  $\delta$  exceeds the critical value  $\delta_c$ , in accordance with the adopted solid body model, the opposite crack surfaces will not interact. That means that the fracture in the plate will continue to increase, developing the destructive process. Thus the critical load ( $\sigma_c$ ) represents a  $\sigma_{\infty}$  value whereby  $\delta=\delta_c$ . We then find from (18) that

$$\sigma_c = \frac{2}{\pi} \sigma_d \arccos \exp\left(-\frac{\pi E \delta_c}{8l \sigma_d}\right). \quad (19)$$

With  $\sigma_c \ll \sigma_d$ , we find from the last formula that

$$\sigma_c = \sqrt{\frac{E \sigma_d \delta_c}{\pi l}}. \quad (20)$$

With reference to brittle materials, we find from (20), by defining  $\delta_c$  on the basis of formula (2), that

$$\sigma_c = \sqrt{\frac{2ET}{\pi l}}. \quad (21)$$

that is, we obtain the known Griffith formula (ref. 3).

With  $l \rightarrow 0$ , formula (21) produces infinitely high  $\sigma_c$  values. There is no such drawback in formula (19) which leads to the conclusion that with  $l \rightarrow 0, \sigma_c \rightarrow \sigma_d$ , i. e., that the strength of a plate with a "zero" fracture equals the strength of an unblemished plate. This is a trivial physical result which does not affect the contemporary generalizations of Griffith's theory. A more complete concept of these generalizations can be formed on the basis of G. I. Barenblatt's studies (ref. 4).

Inasmuch as the destructive process apparently begins with the formation of small cracks in the area of critical concentrations of tension, the above-mentioned theories are unsuitable for an investigation into the origin of the fractures as they deal with infinitely high values of strength. At the same time, it follows from formula (19) that very small cracks produce an insignificant reduction in the strength of the plate.

Formulas (19) and (20) can be applied not only to brittle bodies but also to bodies in which the destructive process is accompanied by plastic deformations. In this case the product  $\sigma_d \delta_c$  indicates the energy used in producing two surfaces

with a single area by well-developed fractures in such a body. Designating  $\sigma_d \delta_c$  by A, we will write formulas (19) and (20) as follows:

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$$\sigma_c = \frac{2}{d} \sigma_c \arccos \exp \left( -\frac{dEA}{8l\sigma_c^2} \right) \quad (22)$$

$$\sigma_c = \sqrt{\frac{EA}{d l}} \quad (\sigma_c \ll \sigma_d) \quad (23)$$

The latter formula, proposed by Orowan (ref. 5), is valid when  $\sigma_c \ll \sigma_d$ .

Just like (19), formula (22) holds true for any  $\sigma_c$  values even when  $\sigma_c$  is of the same order as  $\sigma_d$ .

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Translated for the National Aeronautics and Space Administration  
by John F. Holman and Co. Inc.  
WASHINGTON, D.C. 20037  
NASw-1495